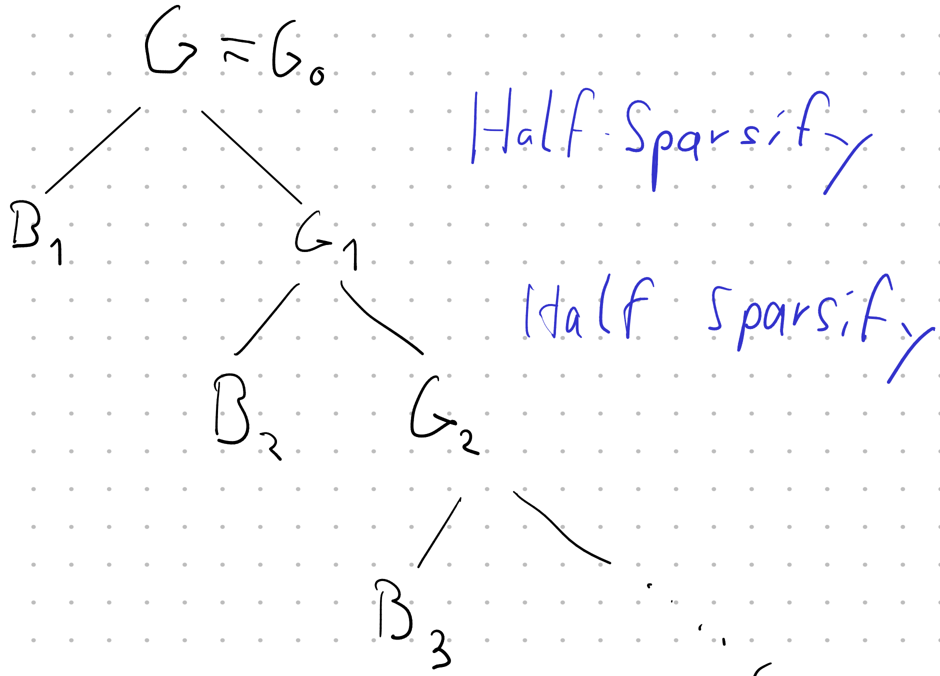
Cut Sparsifier Construction

We saw last time:

<u>Theorem:</u> The Half-Sparsify algorithm constructs, with high probability ($\geq 1 - \frac{1}{n^{c+2}}$), a $(1 \pm \varepsilon)$-cut sparsifier $H = B \cup G'$ of an unweighted graph G of size $|E(B)| = O\left(\frac{n \log n}{\varepsilon^2}\right)$ and $|E(G')| \leq \frac{1}{2} |E(G)|$.

Repeated Sparsification



G_r with $|E(G_r)| = 0$

For $i \geq 1$, the graphs G_i are weighted, but all edges have the same weight. Thus we can apply Half-Sparsify.

The final sparsifier is $H = B_1 \cup B_2 \cup \dots \cup B_t$

<u>Analysis</u>

At each step, • $B_{i+1} \cup G_{i+1}$ is a $(1 \pm \varepsilon)$-sparsifier of G_i, i.e.,

for all cuts $(S, V \setminus S)$

$$(1-\varepsilon) \cdot w_{G_i}(E(S, V\setminus S)) \overset{(1)}{\leq} w_{B_{i+1} \cup G_{i+1}}(E(S, V \setminus S)) \overset{(2)}{\leq} (1+\varepsilon)\, w_{G_i}(E(S, V\setminus S))$$

(3) • $E(|B_{i+1}|) = O\left(\frac{n \log n}{\varepsilon^2}\right)$

(4) • $E(|G_{i+1}|) \leq \frac{1}{2} \cdot E(|G_i|)$

with probability $\geq 1 - \frac{1}{n^{c+2}}$

There are $t \leq n$ iterations, and thus these properties hold with probability $\geq 1 - \frac{n}{c+1}$ in all iterations.

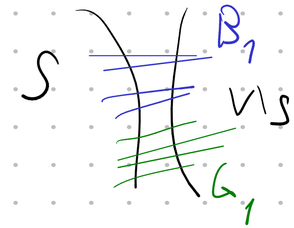
<u>Size:</u>

By (4) $t = O(\log n)$ and thus H has size $O\left(t \cdot \overset{\text{by (3)}}{\frac{n \log n}{\varepsilon^2}}\right) = O\left(\frac{n (\log n)^2}{\varepsilon^2}\right)$

<u>Quality</u>

Consider any cut $(S, V \setminus S)$

$$|E_G(S,V\setminus S)| = w_{G_0}(E_{G_0}(S,V\setminus S)) \underset{\text{by (1)}}{\leq} \frac{1}{1-\varepsilon} \cdot w_{B_1 \cup G_1}(E_{B_1 \cup G_1}(S,V\setminus S))$$



$$= \frac{1}{1-\varepsilon}\Big(w_{B_1}(E_{B_1}(S,V\setminus S)) + \underbrace{w_{G_1}(E_{G_1}(S,V\setminus S))}_{\text{use (1) again}}\Big)$$

$$\leq \frac{1}{1-\varepsilon}\Big(\overset{\frac{1}{1-\varepsilon}\,\checkmark}{w_{B_1}}(E_{B_1}(S,V\setminus S)) + \frac{1}{1-\varepsilon}\big(w_{B_2}(E_{B_2}(S,V\setminus S)) + w_{G_2}(E_{G_2}(S,V\setminus S))\big)\Big)$$

$$= \left(\frac{1}{1-\varepsilon}\right)^2 \cdot \Big(w_{B_1}(E_{B_1}(S,V\setminus S)) + w_{B_2}(E_{B_2}(S,V\setminus S)) + w_{G_2}(E_{G_2}(S,V\setminus S))\Big)$$

$$\vdots$$

$$= \left(\frac{1}{1-\varepsilon}\right)^t \Big(\underbrace{w_{B_1}(E_{B_1}(S,V\setminus S)) + \dots + w_{B_t}(E_{B_t}(S,V\setminus S))}_{= w_H(E_H(S,V\setminus S))} + \overbrace{w_{G_t}(E_{G_t}(S,V\setminus S))}^{=0}\Big)$$

$$= \left(\frac{1}{1-\varepsilon}\right)^t \cdot w_H(E_H(S,V\setminus S))$$

Because $(1+\varepsilon) \leq \frac{1}{1-\varepsilon}$

$$\Rightarrow w_H(E_H(S,V\setminus S)) \geq (1-\varepsilon)^t \cdot |E_G(S,V\setminus S)|$$
$$\geq (1-t\varepsilon) \cdot |E_G(S,V\setminus S)|$$

Bernoulli's Inequality

Similarly $w_H(E_H(S, V\setminus S)) \leq \underbrace{(1+\varepsilon)^t}_{} \cdot |E_G(S, V\setminus S)|$

$$\leq \left(\frac{1}{1-\varepsilon}\right)^t = \frac{1}{(1-\varepsilon)^t} \leq \frac{1}{1-t\varepsilon} \leq 1+2t\varepsilon$$

Because $(1+\varepsilon) \leq \frac{1}{1-\varepsilon} \leq 1+2\varepsilon$ for $0<\varepsilon\leq\frac{1}{2}$

$\rightarrow$ Run the whole algorithm with $\varepsilon' = \dfrac{\varepsilon}{2t}$ to get a $(1\pm\varepsilon)$-cut sparsifier H of G

The size of H then becomes $O\left(n\dfrac{(\log n)^2}{(\varepsilon')^2}\right) = O\left(\dfrac{n(\log n)^4}{\varepsilon^2}\right)$